Graph Theory and its Applications

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Talk organized by MME, AME and SMS

9 May 2025

Outline

1. Birth of Graph Theory
2. Practical Applications
3. School timetable scheduling problem
4. Applications to education
5. Application to $1^k + 2^k + \cdots + n^k$

Birth of Graph Theory

A park in Singapore

Two islands and six bridges

There are two islands within Jurong Lake: Chinese garden and Japanese garden.

There are six bridges in Jurong lake between these two islands or connecting them to the lake bank:

- two bridges between these two islands;
- two bridges between each of these two islands and the lake bank.

A puzzle

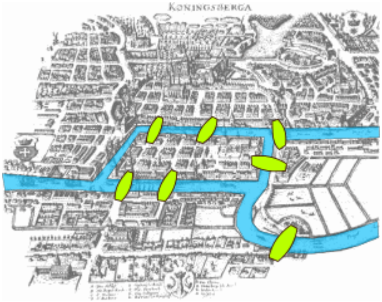
Puzzle: *Is there a walking path that can cross each of the six bridges in Jurong lake* **once and only once**?

Answer: Yes


Lakeside Field
Ficus Lawn
Butterfly Field
Jurong Town Hall
Yuan Ching Rd
The Oval

The Königsberg bridge problem

Königsberg bridge problem, a recreational mathematical puzzle, set in the old Prussian city of Königsberg.

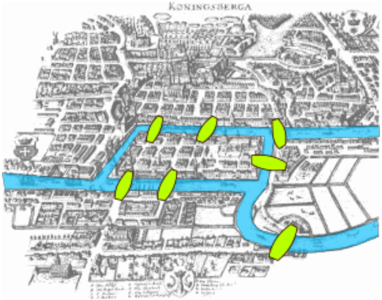


1525 – 1871: Königsberg was a city of Prussia;

1871 – 1945: Königsberg was a city of German after Prussia formed the German Empire.

1945 – now: It became a city of Russia after World War II, and was renamed as Kaliningrad.

The Königsberg bridge problem

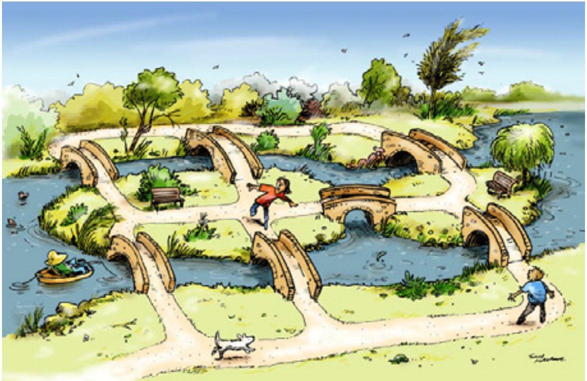


The city of Königsberg set on both sides of the Pregel River, and included **two islands** which were connected to each other, and to the two mainland portions of the city, by **seven bridges** .

The people of Königsberg amused themselves by **trying to devise a walking path around their city** which would **cross each of these seven bridges once and only once** and return to their starting point.

The Königsberg bridge problem

The Königsberg bridge problem in 1800s: *to devise a walk through the city that would cross each of those bridges once and only once and finally return to the original point.*



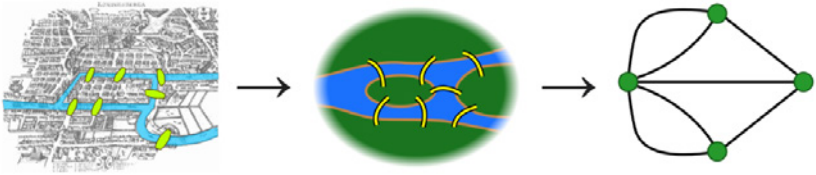
Settled by Leonhard Euler in 1736

Leonhard Euler (1707 – 1783) was a Swiss mathematician who made enormous contributions to a wide range of mathematics and physics including analytic geometry, trigonometry, geometry, calculus and number theory.

Euler is regarded as arguably **the most prolific contributor** in the history of mathematics and science, and **the greatest mathematician** of the 18th century.

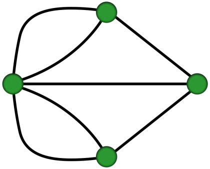
Applying Graph Theory

Step 1: Transfer it to a problem on a graph.



Step 2: Check if the graph has a trail which goes through each edge once and only one, and finally returns to the starting vertex.

Euler's conclusion



There is no trail in this graph which crosses each edge exactly once, and finally returns to the starting vertex.

Theorem (Euler (1736))

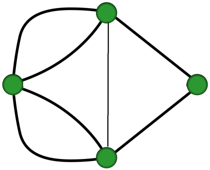
If a graph is **connected**, *then*

there exists a trail in this graph which crosses each edge exactly once, and finally returns to the starting vertex

$$\Updownarrow$$

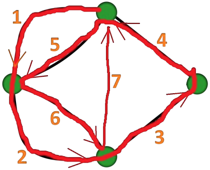
the graph has **no vertices with odd degrees**

Example



Problem: In this revised graph, is there a trail which crosses each edge exactly once, and finally returns to the starting vertex?

Note that each vertex in this graph has an even degree.



There exists such a trail in this graph.

Euler pioneered a new field

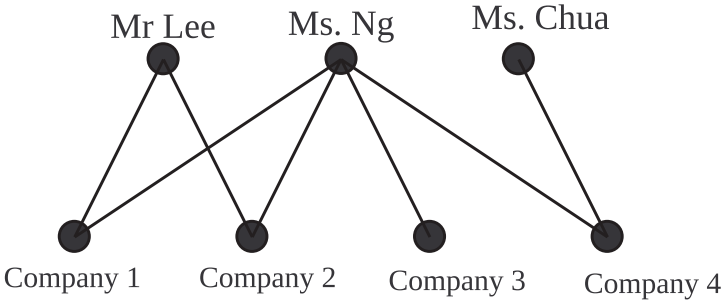
In Euler's paper, he

- represented the landmasses as vertices and the bridges as edges; and
- introduced key concepts such as degree (the number of edges incident with a vertex), laying the groundwork for the field of graph theory.

His paper was the first formal result in graph theory, marking **the birth of the subject** and introducing ideas that are foundational to both graph theory and topology.

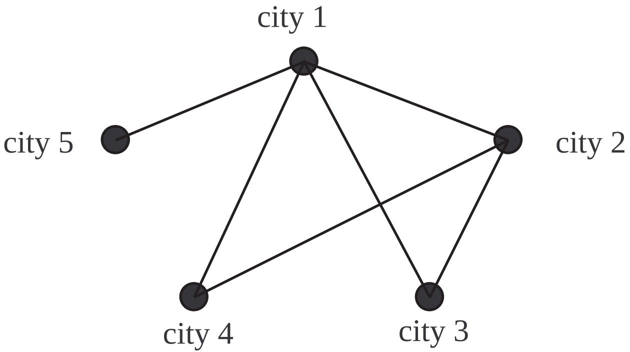
Example 1

1. The following graph models job-applications:

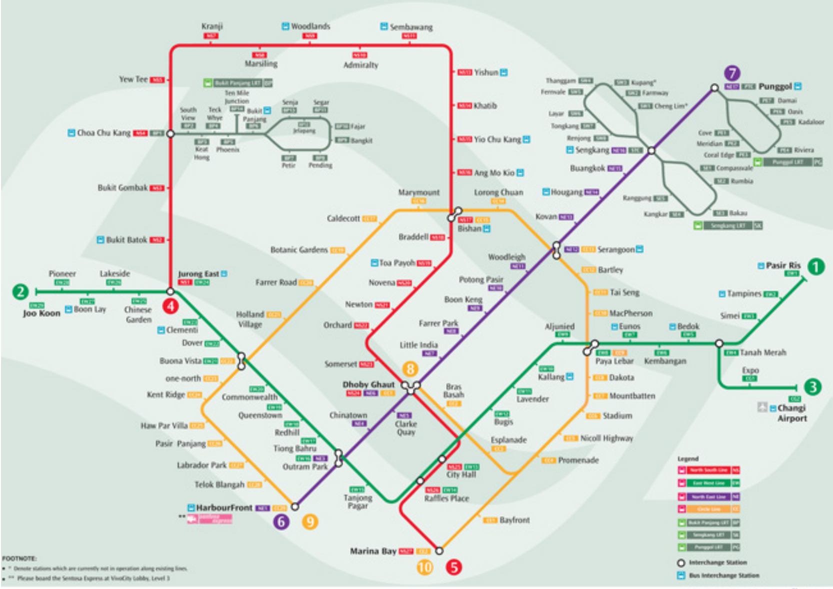


Example 2

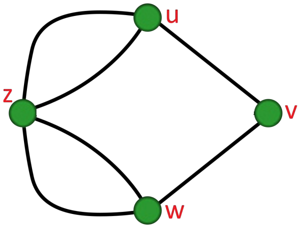
1. The following graph models pairs of cities with direct flights between them:



MRT map can be transferred into a graph



Introduction of Graphs

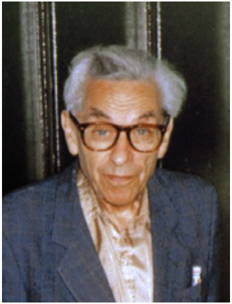


This graph has four vertices and six edges.

A graph consists of dots and lines each of which connects two dots.

In graph theory, a dot is called a vertex and a line is called an edge.

Paul Erdős (1913 – 1996)



Paul Erdős was a Hungarian mathematician.

He pursued and proposed problems in many mathematical subjects, such as discrete mathematics, graph theory, number theory, etc.

Much of his work centered on discrete mathematics, cracking many previously unsolved problems in the field.

He was **one of the most prolific and influential mathematicians of the 20th century.**

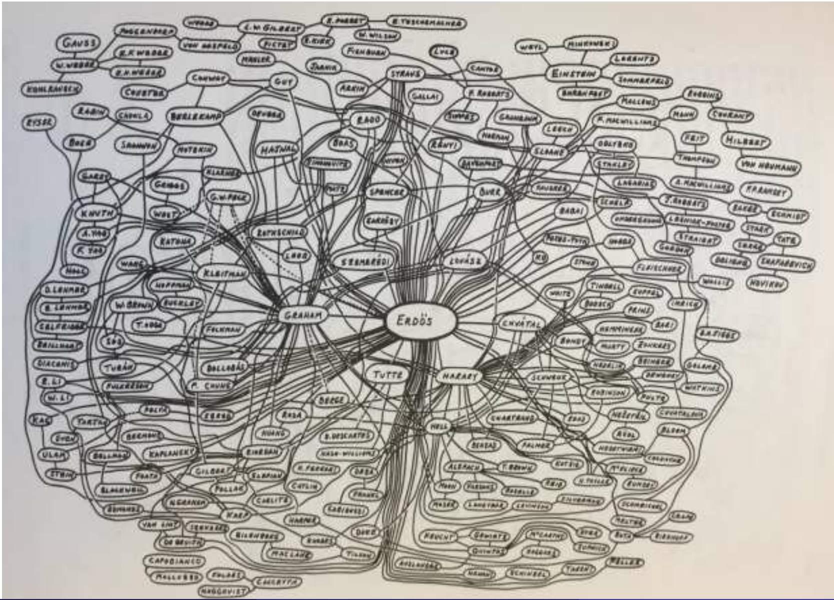
Erdős number

Erdős considered mathematics to be a social activity and often collaborated on his papers, having 511 joint authors , many of whom also have their own collaborators.

The Erdős number measures the “collaborative distance" between a mathematician and Erdős, determined by the minimum number of steps it takes to connect them through co-authorship of mathematical papers.

Thus, his direct 511 co-authors have Erdős number one, and theirs, more than 12500 people, have Erdős number two, and so forth.

The Erdős collaboration graph



William Thomas Tutte (1917 – 2002)

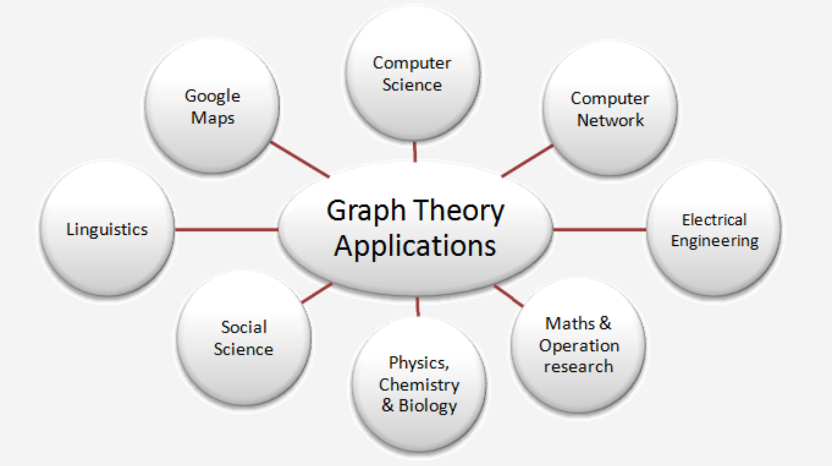
He was a British-born Canadian code breaker and mathematician, retiring from the University of Waterloo.

- During the Second World War, he made a fundamental advance in cryptanalysis of the Lorenz cipher, a major Nazi German cipher system.
- He also made **significant mathematical accomplishments**, including foundation work in the fields of **graph theory** and **matroid theory**, dominating these areas for around three decades.
- He was awarded **the Order of Canada**, an honor bestowed upon him in 2000 as an **Officer of the Order**.

Practical Applications

Applications of Graph Theory

Real Life Applications of Graph Theory



Applications that will be introduced

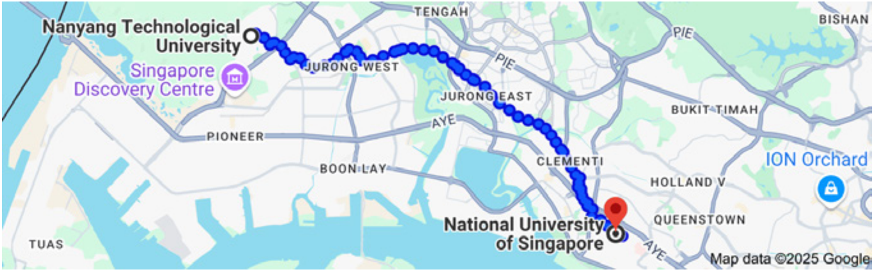
- Shortest path problem;
- Shortest network problem;
- Traveling salesman problem; and
- Chinese postman problem.

Google Search for shortest distance

Problem: *How can we find* **a shortest path** *between any two given points on earth?*

Example: *What is a* **shortest way** *to walk from NTU to NUS?*

Such a shortest walking path can be found easily by Google, but not easy without using any searching engine.



Transferred into a problem in graphs

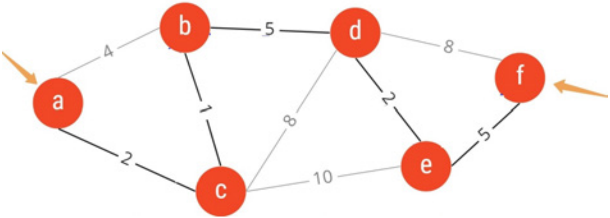
The problem of finding a shortest distance between two places can be transferred into a problem on a weighted graph.

Junctions, Locations $\Longrightarrow$ vertices
Roads, streets $\Longrightarrow$ weighted edges
(weights can be distance, time, traffic, etc.).

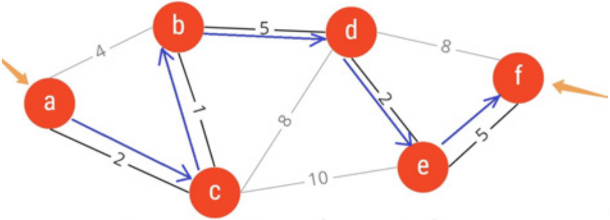
The problem on a weighted graph can be solved by some algorithms. Dijkstras Algorithm is one of them.

Example

Problem: *Find a shortest path from a to f in the following graph.*



The shortest distance from a to f is $2 + 1 + 5 + 2 + 5 = 15$.



The Shortest Network Problem

Connector Problem

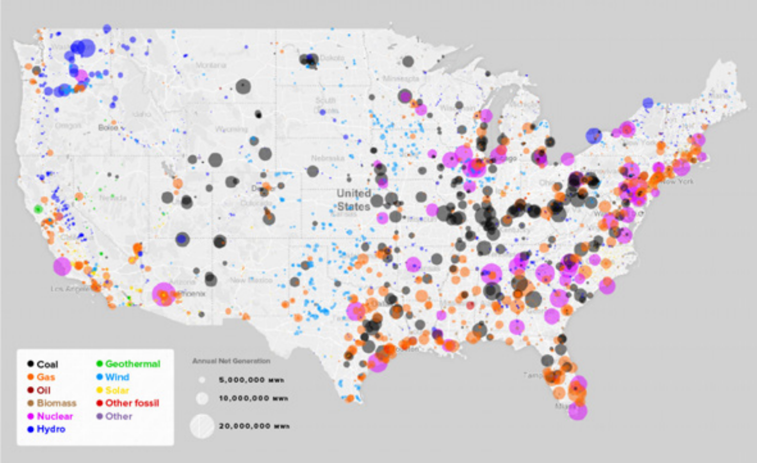
Given a set of locations (or nodes) on the earth, how can we seek a shortest possible way to connect them, often subject to certain constraints?

Constrains: cost, distance, traffic, capacity, etc.

Example: *How can we identify an optimal (shortest) water pipeline network layout in Singapore?*

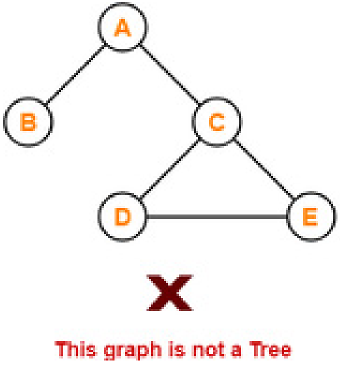
Example

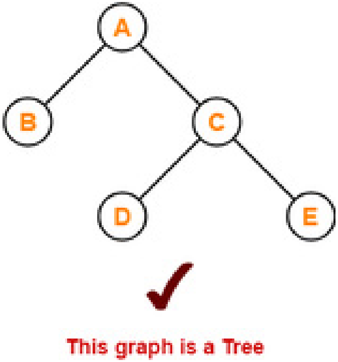
Problem: *How can people find a shortest network connecting all power stations in US?*



Trees: a special type of graphs

Definition: *In graph theory, a tree is a connected graph which has no cycles.*





Trees: a special type of graphs

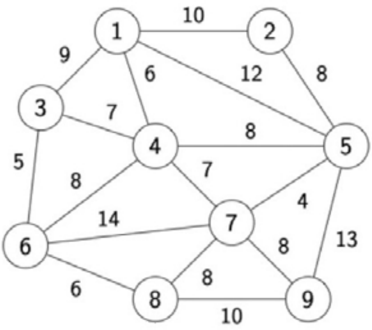
A real tree is connected. There were hardly any trees with circles.

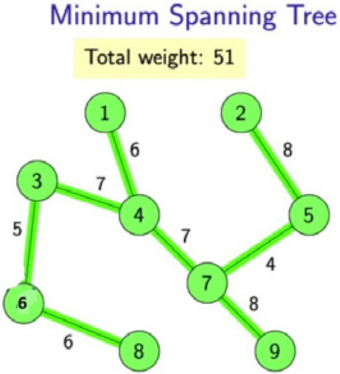
But the following tree is exceptional.

Transferred into a problem in weighted graphs

Finding a shortest network is transferred into the problem of finding a **minimum spanning tree** in a weighted graph.

Problem: *Find a minimum spanning tree in this weighted graph.*





Traveling Salesman Problem

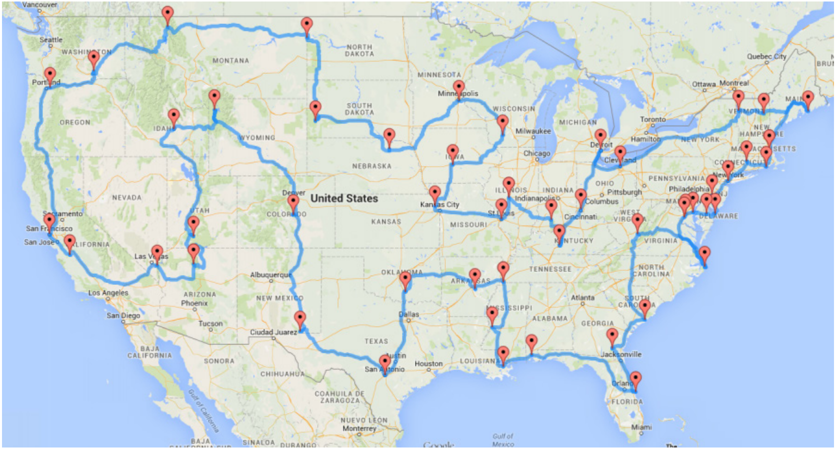
Problem: *A traveling salesman wishes to visit a number of towns and then return to his starting point. Given the travelling times between towns, how should he plan his itinerary so that he visits each town exactly once and minimizes his total traveling time?*

Example: *Assume that a school bus needs* **to pick up students from their homes and drop them off at RI, RJC, HCI, and HCJC.** *How should the driver plan the route* **to visit each location, including the students' homes, exactly once** *while minimizing the total travel time?*

The above problem can be used as a school project.

Example

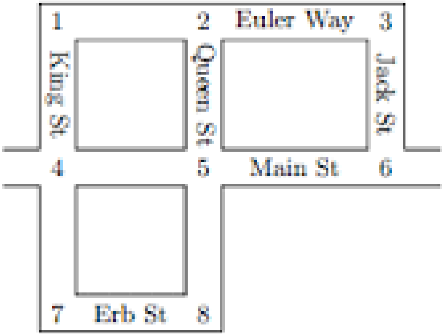
A route to visit many cities and return to the original position.

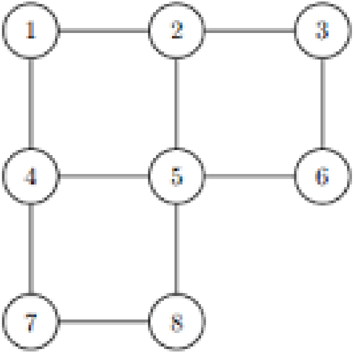


A street map $\Longrightarrow$ a graph

road junctions $\Longrightarrow$ vertices

streets $\Longrightarrow$ edges



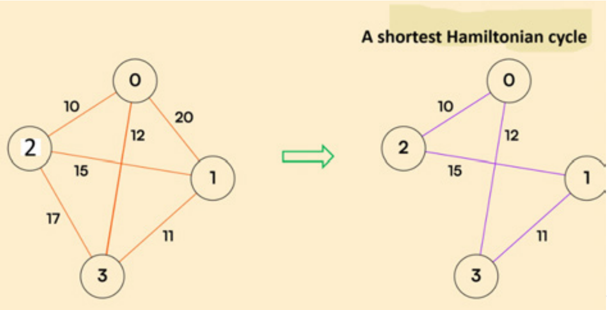


Transferred into a problem in graphs

A **Hamitonian cycle** *in a graph is a cycle which contains all vertices in this graph.*

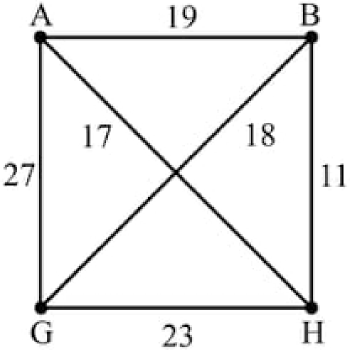
The traveling salesman problem can be transferred into:

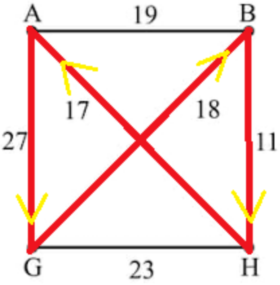
Problem: *Find* **a shortest Hamiltonian cycle** *in a weighted graph.*



A puzzle

Puzzle: *Find* **a shortest Hamiltonian cycle** *in the following weighted graph.*





A shortest Hamiltonian cycle: $27 + 17 + 11 + 18 = 73$.

Chinese Postman Problem

Problem: *A postman picks up mail at the post office, delivers it, and then returns to the post office. He must go through each street at least once. Subject to this condition, how can he choose a route entailing as little walking as possible?*

The problem was originally studied by the Chinese mathematician Meigu Guan in 1960, whose Chinese paper was translated into English in 1962. The original name "Chinese postman problem" was coined in his honor.

Transferred into a graph problem

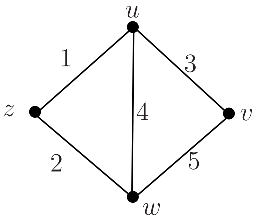
Example: *Suppose a garbage truck is responsible for* **collecting garbage from all the streets in West Coast**, *how should the driver arrange the route?*

Problem in a connected graph:
Find a shortest closed walk in a connected graph so that it goes through every edge in this graph at least once.

Example

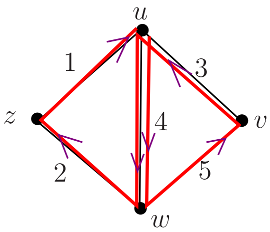
Problem

Find a shortest closed walk in the following weighted graph so that it goes through each edge at least once.



The weight of each edge can be considered as the distance.

Attempt



$u \to w \to v \to u$

$\to w \to z \to u$

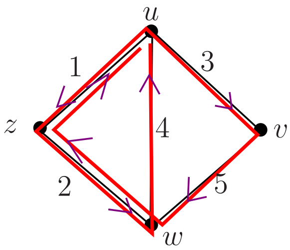
The total distance of this closed walk is

$$4 + 5 + 3 + 4 + 2 + 1 = 19.$$

Is it shortest?

It is not a shortest closed walk among all such closed walks.

Attempt



$u \to z \to w \to u \to v \to w \to z \to u$

This closed walk repeating edges uz and zw has its total distance 18:

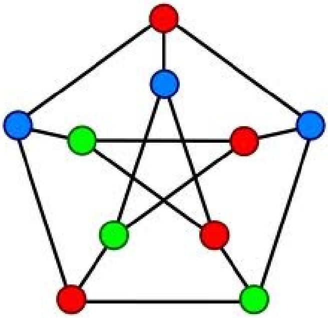
$$1 + 2 + 5 + 3 + 1 + 2 + 4 = 18.$$

It is **shortest** among all such closed walks.

School timetable scheduling problem

Proper coloring

For a positive integer k, a (proper) k-coloring of a graph G is a way of assigning k colors to vertices in G, one color for each vertex, such that any two adjacent vertices are assigned different colours.



Any two vertices assigned with the same color are not adjacent.

An example

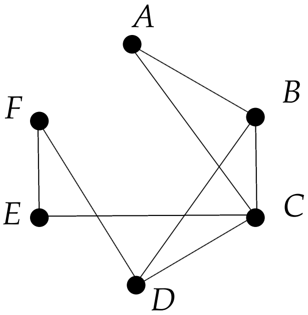
- A tropical fish hobbist had six different types of fish: Alphas, Betas, Certas, Deltas, Epsalas, and Fetas, designated by A, B, C, D, E, and F, respectively.
- Because of predator-prey relatinships, water conditions, and size, some fish cannot be kept in the same tank:

Type	A	B	C	D	E	F
not with	B,C	A, C, E	A,B, D, E	C,F	B,C,F	D,E

Problem: *What is the minimum number of tanks needed to keep all the fish?*

An example

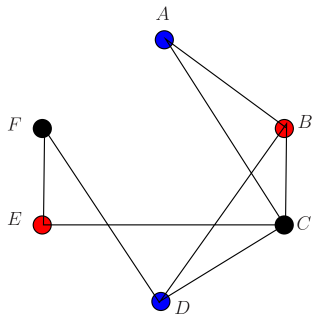
Type	A	B	C	D	E	F
not with	B,C	A, C, E	A,B, D, E	C,F	B,C,F	D,E



Vertex A represents type A of fish.

In this graph, any two vertices are joined by an edge if the fish of the types they represented cannot be together.

An example



At least three colors are needed for a proper coloring.

Types A and D can be together, types B and E can be together, and types C and F can be together.

Hence **at least three tanks** are needed to keep all six types of fish.

Tank 1: Types A and D;
Tank 2: Types B and E;
Tank 3: Types C and F.

Timetable scheduling problem

The timetable scheduling problem is a classic example of a combinatorial optimization problem.

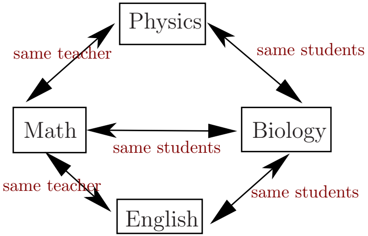
Problem

The problem is to create a timetable for a school that schedules all lessons over a fixed period (e.g., a week), while meeting the needs of teachers and students that

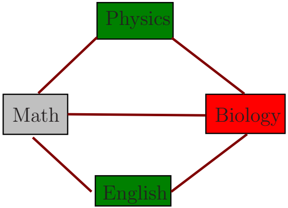
- a teacher can teach one class only at a time; and
- a student can attend one lesson only at a time.

Example

Teaching arrangements



Modeled by a graph with a proper 3-coloring



A suitable timetable

9:00-10:00	Physics	English
10:00-11:00	Math	
11:00-12:00	Biology	

Timetable scheduling problem

- Assume that a school has 100 different courses: $C_1, C_2, \cdots, C_{100}$.
- Let G be the graph with vertices $C_1, C_2, \cdots, C_{100}$ such that C_i and C_j are joined by an edge if the courses they represented are either taught by some teacher or taken by some student.
- Then, find a proper coloring in G, and the modules assigned the same color can be arranged in the same time-slot.

Other applications of vertex-coloring

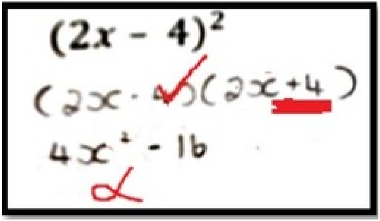
- exam time-table scheduling;
- Sports scheduling;
- Task Assignment in Parallel Computing;
- Frequency Assignment (Wireless Communication);
- Chemical Storage & Safety;
- Air Traffic Flow Management;
- Biological Networks (DNA Sequencing);
- etc.

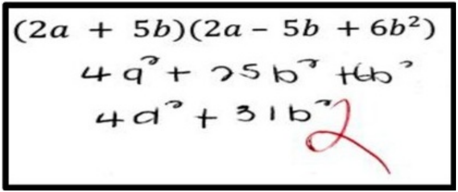
Applications to education

Topics in this section

- Applications in algebraic expansion;
- Social Network Analysis;
- Knowledge graphs;
- Reasoning Graphs (or reason structure graph).

Mistakes in algebraic expansion



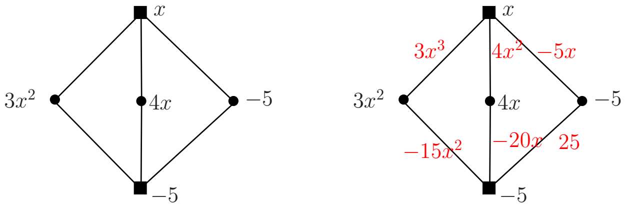


$$(x+y)^2 = x^2 + y^2; \qquad (a+b+c)^2 = a^2 + b^2 + c^2;$$

$$(x-y)^2 = x^2 - y^2. \qquad (a+b-c)^2 = a^2 + b^2 - c^2.$$

Apply graphs in algebraic expansion

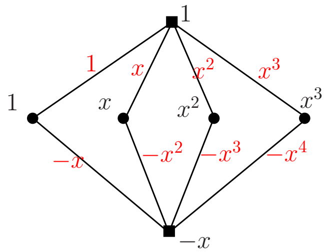
$$\text{Expand } (x-5)(3x^2+4x-5) = ?$$



$$3x^3+4x^2-5x-15x^2-20x+25 = 3x^3-11x^2-25x+25.$$

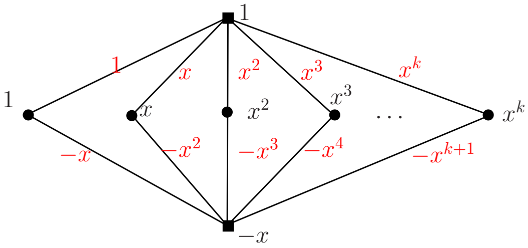
Example

$$(1-x)(1+x+x^2+x^3) = ?$$



$$\begin{aligned}(1-x)(1+x+x^2+x^3) &= 1+x+x^2+x^3-x-x^2-x^3-x^4 \\ &= 1-x^4.\end{aligned}$$

$(1-x)(1+x+x^2+\cdots+x^k) = ?$



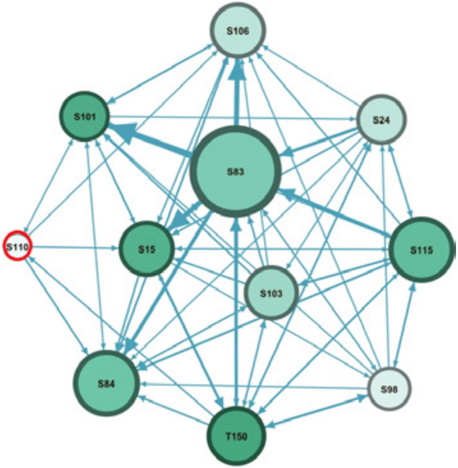
$$
\begin{aligned}
& (1-x)(1+x+x^2+x^3+\cdots+x^k) \\
= \;& 1+x+x^2+x^3+\cdots+x^k - x - x^2 - x^3 - x^4 - \cdots - x^{k+1} \\
= \;& 1 - x^{k+1}.
\end{aligned}
$$

Theorem

$$(1-x)(1+x+x^2+\cdots+x^k) = 1 - x^{k+1}.$$

Social Network Analysis

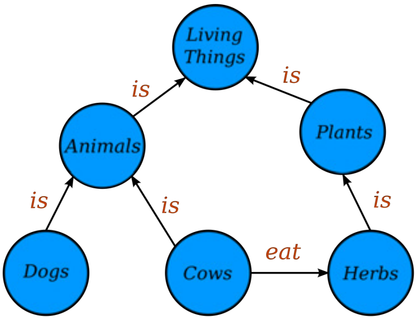
What is a social network analysis?



A directed graph showing the patterns of the interactions among a group of students and their tutor.

Student S110 is most inactive and might need more attention from the tutor.

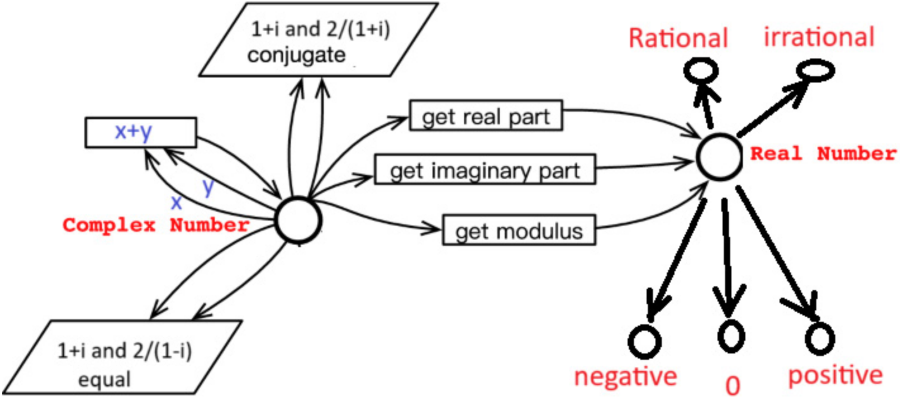
Knowledge graph



A knowledge graph uses a graph-structured data model to represent and operate on data.

Knowledge graphs are often used to store interlinked descriptions of objects, events and situations.

A knowledge graph on numbers



Steps: $$\frac{2}{x-2} - \frac{1}{x+2} = \frac{3}{x^2-4};$$

✓ $$\frac{2(x^2-4)}{x-2} - \frac{1(x^2-4)}{x+2} = \frac{3(x^2-4)}{x^2-4};$$ de-denominator

✓ $$\boxed{2(x+2)-(x-2)=3;}$$ linearization

$$2x+4-x+2=3;$$

$$x+6=3;$$

✓ $$\boxed{x=-3.}$$ potential roots

✓ $$\boxed{\text{when } x=-3,\ x^2-4\neq 0.}$$ verify

Hence $x=-3$ **is the only solution**.

A reasoning graph

de-denominator

↓

transfer to a polynomial equation

↓

determine potential roots

↓

verify roots

a graph showing the operations of the key steps in the solution.

A reasoning graph

A **reasoning graph** (or **reasoning structure graph**) for a solution is a visual representation that breaks down the logical flow and dependencies between steps in solving the problem.

It helps illustrate how different mathematical concepts, theorems, and reasoning steps are connected to reach the final solution.

Prove that $\sqrt{2}$ is irrational

Suppose that $\sqrt{2}$ is rational. Then there are positive integers p and q, where $GCD(p,q)=1$, such that $\sqrt{2}=\frac{p}{q}$.

✓ $\boxed{\sqrt{2}=\frac{p}{q}}$ $\Rightarrow$ $2=\frac{p^2}{q^2}$;

$\Rightarrow$ $2q^2=p^2$;

✓ $\Rightarrow$ $\boxed{p \text{ is even}}$;

$\Rightarrow$ Let $p=2k$, $k\in\mathbb{N}$. Then $2q^2=4k^2$;

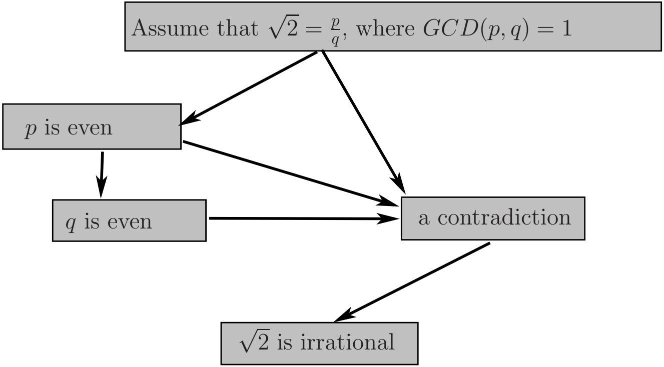
$\Rightarrow$ $q^2=2k^2$;

✓ $\Rightarrow$ $\boxed{q \text{ is even}}$;

✓ $\Rightarrow$ $\boxed{\textbf{a contradiction with } GCD(p,q)=1}$.

Hence $\sqrt{2}$ **is irrational**.

A reasoning graph of the proof



Can be applied to prove that $\sqrt{3}$ is irrational.

Applying the above reasoning graph, one can prove that:
d is prime $\Longrightarrow$ $\sqrt{d}$ **is irrational**.

Importance of a reasoning graph

- **Clarity**: Helps visualizing logical dependencies.
- **Debugging**: Identifies missing or incorrect steps.
- **Learning**: Reinforces understanding of problem-solving strategies.
- **Generalization**: Can be adapted to similar problems.

Application to $1^k + 2^k + \cdots + n^k$

How to find a formula for $1^k + 2^k + \cdots + n^k$

$1 + 2 + 3 + \cdots + 100 = ?$ Easy

$1^2 + 2^2 + 3^2 + \cdots + 100^2 = ?$ Not easy

$1^3 + 2^3 + 3^3 + \cdots + 100^3 = ?$ **Challenging**

Problem

For given positive integers k and n, how can we apply graph theory and combinatorics to find a formula for

$$1^k + 2^k + \cdots + n^k?$$

Sum $1+2+3+\cdots+100$

$$1+2+3+\cdots+100 = \frac{1+100}{2} \times 100 = \frac{100 \times 101}{2}.$$

Any generalization?

$$\boxed{1+2+3+\cdots+100 = \tfrac{100\times 101}{2}}$$

$$\Updownarrow$$

$$\binom{1}{1}+\binom{2}{1}+\binom{3}{1}+\cdots+\binom{100}{1} = \binom{101}{2}$$

What are these numbers $\binom{1}{1}, \binom{2}{1}, \binom{3}{1}, \cdots, \binom{100}{1}, \binom{101}{2}$?

These numbers are **numbers of combinations**.

Definition of a combination

- A combination of n items taken r at a time ($n \geq r$) is a selection of r items from a set of n different members, such that the order of selection does not matter.
- **Example**: There are four combinations of $\{1, 2, 3, 4\}$ taken 3 at a time:

$$\{1, 2, 3\},\ \{1, 2, 4\},\ \{1, 3, 4\},\ \{2, 3, 4\}.$$

- Let $\binom{n}{r}$, or $C(n, r)$, denote **the number of combinations of n items taken r at a time**. Thus,

$$\binom{4}{3} = 4.$$

Formula for $\binom{n}{2}$, i.e., $C(n,2)$

- $\boxed{\binom{4}{2} = 6}$, since $\{1,2,3,4\}$ has exactly six combinations taken 2 at a time:

$$\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}.$$

- Observe that

$$\binom{4}{2} = 6 = \frac{4 \times 3}{2}.$$

- Generally, for any integer $n \geq 0$,

$$\binom{n}{2} = \frac{n(n-1)}{2}.$$

Formula for $\binom{n}{r}$

For any positive integers n and r,

$$\binom{n}{r} = \frac{n(n-1)\cdots(n-r+1)}{r \times (r-1) \times \cdots \times 1}, \quad \text{or} \quad \binom{n}{r} = \frac{n!}{r!(n-r)!},$$

where $r! = r \times (r-1) \times \cdots \times 1.$

For example,

$$\binom{5}{3} = \frac{5 \times 4 \times 3}{3 \times 2 \times 1} = 10$$

or

$$\binom{5}{3} = \frac{5!}{3! \times 2!} = \frac{5 \times 4 \times 3 \times {\color{blue}2 \times 1}}{{\color{red}3 \times 2 \times 1} \times {\color{blue}2 \times 1}} = 10.$$

Applied in Binomial Theorem

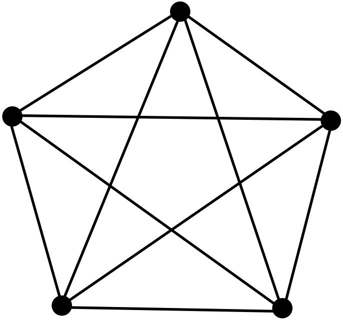
Binomial Theorem

For any real numbers x and y,

$$\begin{aligned}
(x+y)^2 &= \binom{2}{0}x^2 + \binom{2}{1}xy + \binom{2}{2}y^2; \\
(x+y)^3 &= \binom{3}{0}x^3 + \binom{3}{1}x^2y + \binom{3}{2}xy^2 + \binom{3}{3}y^3; \\
(x+y)^n &= \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n}y^n,
\end{aligned}$$

$$\forall n = 1, 2, 3, \cdots .$$

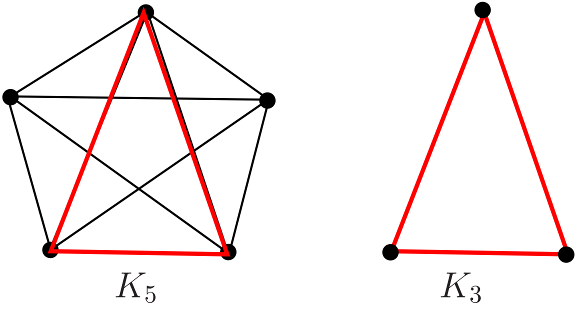
Counting subgraphs in K_5



K_5

K_n denotes the complete graph with n vertices in which every pair of vertices are adjacent.

Counting subgraphs in K_5



K_5 contains $\binom{5}{2} = \frac{5\times4}{2\times1} = 10$ edges.

K_5 contains $\binom{5}{3}$ subgraphs isomorphic to K_3:

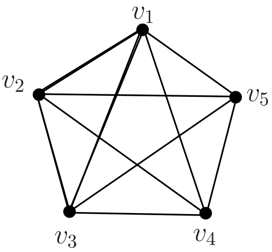
$$\binom{5}{3} = \frac{5 \times 4 \times 3}{3 \times 2 \times 1} = 10.$$

Interpret an identity by a graph

$$\binom{5}{2} = \binom{1}{1} + \binom{2}{1} + \binom{3}{1} + \binom{4}{1}$$

i.e.,

$$10 = 4 + 3 + 2 + 1.$$



$\binom{5}{2}$ is the total number of edges in K_5.

$\binom{4}{1}$ edges contain v_1: $\{v_1v_2, v_1v_3, v_1v_4, v_1v_5\}$.

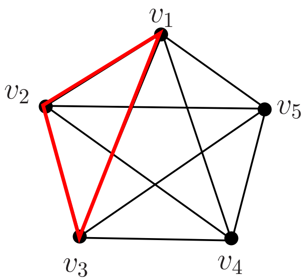
$\binom{3}{1}$ edges contain v_2, but not v_1: $\{v_2v_3, v_2v_4, v_2v_5\}$.

$\binom{2}{1}$ edges contain v_3, but not v_1 nor v_2: $\{v_3v_4, v_3v_5\}$.

$\binom{1}{1}$ edges contain v_4 and v_5: $\{v_4v_5\}$.

Interpretation for the identity

Explain $$\binom{5}{3} = \binom{4}{2} + \binom{3}{2} + \binom{2}{2}.$$



K_5 has $\binom{5}{3}$ subgraphs which are isomorphic to K_3.

$\binom{4}{2}$ of them contains vertex v_1.

$\binom{3}{2}$ of them contains vertex v_2 but v_1.

$\binom{2}{2}$ of them contains vertex v_3 but v_1 and v_2.

An important property

Property

For any integers n, r with $n \geq r \geq 1$,

$$\binom{n+1}{r+1} = \binom{n}{r} + \binom{n-1}{r} + \cdots + \binom{r}{r}.$$

For example, for $n = 100$, $r = 1, 2$, we have

$$\binom{101}{2} = \binom{100}{1} + \binom{99}{1} + \cdots + \binom{1}{1};$$

$$\binom{101}{3} = \binom{100}{2} + \binom{99}{2} + \cdots + \binom{2}{2}.$$

Find a formula for $1^2 + 2^2 + \cdots + n^2$

$$k^2 = k(k-1) + k = 2 \times \frac{k(k-1)}{2} + k = 2\binom{k}{2} + k.$$

Thus,

$$\begin{aligned}
1^2 &= 2\binom{1}{2} + 1 = 1; \\
2^2 &= 2\binom{2}{2} + 2; \\
3^2 &= 2\binom{3}{2} + 3; \\
&\cdots \\
n^2 &= 2\binom{n}{2} + n.
\end{aligned}$$

Sum $1^2 + 2^2 + \cdots + n^2$

$$
\begin{aligned}
& 1^2 + 2^2 + \cdots + n^2 \\
= \;& 2\left(\binom{2}{2} + \binom{3}{2} + \cdots + \binom{n}{2}\right) + 1 + 2 + \cdots + n \\
= \;& 2\binom{n+1}{3} + \frac{n(n+1)}{2} \\
= \;& \frac{2(n+1)n(n-1)}{3 \times 2 \times 1} + \frac{n(n+1)}{2} \\
= \;& \frac{(n+1)n(n-1)}{3} + \frac{n(n+1)}{2} \\
= \;& n(n+1)\left(\frac{n-1}{3} + \frac{1}{2}\right) \\
= \;& \frac{n(n+1)(2n+1)}{6}.
\end{aligned}
$$

Find a formula for $1^3 + 2^3 + \cdots + n^3$

How to find a formula for $1^3 + 2^3 + \cdots + n^3$?

First, we determine constants a, b, c such that for all integers k,

$$k^3 = a\binom{k}{3} + b\binom{k}{2} + ck = \frac{ak(k-1)(k-2)}{3 \times 2 \times 1} + \frac{bk(k-1)}{2 \times 1} + ck.$$

$$\begin{cases} \text{when } \ k = 1, & 1^3 = a\binom{1}{3} + b\binom{1}{2} + c & \Rightarrow c = 1; \\ \text{when } \ k = 2, & 2^3 = a\binom{2}{3} + b\binom{2}{2} + 2 \times 1 & \Rightarrow b = 6; \\ \text{when } \ k = 3, & 3^3 = a\binom{3}{3} + 6\binom{3}{2} + 3 \times 1 & \Rightarrow a = 6. \end{cases}$$

Express k^3

For any integer $k \geq 0$,

$$k^3 = 6\binom{k}{3} + 6\binom{k}{2} + k.$$

Thus,

$$\begin{aligned}
1^3 &= 6\binom{1}{3} + 6\binom{1}{2} + 1; \\
2^3 &= 6\binom{2}{3} + 6\binom{2}{2} + 2; \\
3^3 &= 6\binom{3}{3} + 6\binom{3}{2} + 3; \\
&\cdots \\
\text{for all integer } k > 0, \quad k^3 &= 6\binom{k}{3} + 6\binom{k}{2} + k.
\end{aligned}$$

Applying $k^3 = 6\binom{k}{3} + 6\binom{k}{2} + k$

$$
\begin{aligned}
& 1^3 + 2^3 + \cdots + n^3 \\
= \; & 6\left(\binom{1}{3} + \binom{2}{3} + \cdots + \binom{n}{3}\right) + 6\left(\binom{1}{2} + \binom{2}{2} + \cdots + \binom{n}{2}\right. \\
& + (1 + 2 + \cdots + n) \\
= \; & 6\binom{n+1}{4} + 6\binom{n+1}{3} + \frac{n(n+1)}{2} \\
= \; & \frac{6(n+1)n(n-1)(n-2)}{4 \times 3 \times 2} + \frac{6(n+1)n(n-1)}{3 \times 2} + \frac{n(n+1)}{2} \\
= \; & \frac{(n+1)n(n-1)(n-2)}{4} + (n+1)n(n-1) + \frac{n(n+1)}{2} \\
= \; & \frac{n^2(n+1)^2}{4}.
\end{aligned}
$$

Takeaways

Question 1: *Interpret the identity below by a problem on graphs:*

$$\binom{m+n}{3} = \binom{m}{3} + \binom{m}{2}\binom{n}{1} + \binom{m}{1}\binom{n}{2} + \binom{n}{3}.$$

Question 2: *Prove that* $\sqrt{5}$ *is irrational.*

Question 3: *Find a formula for the following expression:*

$$1^4 + 2^4 + 3^4 + \cdots + n^4.$$

Thanks for your attendance

SPECIAL THANKS to

Department of Mathematics and Mathematics Education, NIE

The Association of Mathematics Educators (AME)

The Singapore Mathematical Society (SMS)

Feedback